ISSN 2393-8471

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 1, pp: (1-18), Month: April 2024 – September 2024, Available at: www.paperpublications.org

Investigation of the Dynamic Behaviour of Functionally Graded Rectangular Plates Resting on Winkler Foundation

Onwuka, D. O[1]., Ohia, C. A[2]., Arimanwa, J. I[3]., Onwuka, S. U[4].

[1,2,3]Civil Engineering Department, Federal University of Technology, Owerri, Nigeria

[4]Department of Project Management, Federal University of Technology, Owerri, Nigeria

DOI: https://doi.org/10.5281/zenodo.10948655

Published Date: 09-April-2024

Abstract: **This study assesses the vibration characteristics of functionally graded rectangular plates resting on Winkler elastic foundation. Integral calculus has been applied to the beam analogy method for the evaluation of the non-dimensional frequency parameters of isotropic functionally graded (FG) rectangular plates resting on Winkler elastic foundation. The fundamental assumptions of linear, elastic, small-deflection theory of bending for thin plates due to Kirchhoff are taken into consideration. Using direct integration, characteristic orthogonal polynomials (COPs) shape function for plates clamped on two opposite edges and simply supported on two other opposite edges (SCSC) is formulated. The effect of aspect ratios on the natural frequency of the plate is examined. The findings of this study show that an increase in aspect ratio results in an increase frequency of the plate. Adding an elastic foundation increases the non-dimensional frequency parameter of the plates. Like plates resting on Winkler foundation, an increase in aspect ratio, causes a corresponding increase in frequency for plates not subjected to the effect of Winkler elastic foundation. It is also observed that an increase in power law index decreases the frequency parameters of the plate. Results are in tandem with those in the literature.**

Keywords: **Free vibration, rectangular plate, Winkler foundation, functionally graded plate, analytical solution.**

I. INTRODUCTION

As a common structural component, rectangular plates have been widely used in aerospace, military and marine industries and other various engineering fields. In the past decades, the problem/analysis of the transverse vibrations of plates has the great attention of several researchers as evidenced by the numerous related research papers on the free transverse vibrations of rectangular plates. Problems involving rectangular plates fall into three distinct categories: plates with all edges simply supported; plates with a pair of opposite edges simply supported; plates which do not fall into any of the above categories. Common applications of plates on an elastic foundation include foundations, storage tanks, swimming pools, floor systems of buildings and highways, airfield pavements, etc.

Functionally graded materials (FGMs) are generally ceramic-metal composites in which material properties vary continuously in the thickness direction from one surface to the other. The ceramic constituent provides high-temperature resistance due to its low thermal conductivity. On the other hand, the ductile metal constituent prevents fracture caused by the stresses due to high temperature gradient in a very short span of time.

Many studies for free vibration analysis of rectangular plates resting on elastic foundation are available in the literature. These studies are done by means of both numerical and analytical approaches. A new version of the differential quadrature method for assessing the vibration characteristics of rectangular plates resting on elastic foundations carrying any number of sprung masses was proposed by Hsu [1]. The first six natural frequencies of plates with various foundation stiffnesses were highlighted. They also analyzed the effect of aspect ratios on the natural frequency of plates on elastic foundation. Using the finite cosine integral transform method, Li et al. [2] presented the analytical solutions for rectangular plates on the Winkler elastic foundation with four edges free. In the analysis, the classical Kirchhoff rectangular plate was considered. Chakraverty and Pradhan [3] investigated the free vibration of functionally graded (FG) rectangular plates subject to different sets of boundary conditions within the framework of classical plate theory. The parametric resonance characteristics of functionally-graded material (FGM) plates on elastic foundation under biaxial in plane periodic load was studied by Ramu and Mohanty [4].

Chakraverty and Pradhan [3], Hosseini-Hashemi et al. [5], Kumar et al. [6], and Sayyad and Ghugal [7] have studied in detail, the free vibration of functionally graded rectangular plates. The bending solutions of free rectangular thin plates, based on the Winkler model, were obtained by a new symplectic superposition method. In a separate study, Bahmyari et al. [8] analysed the free vibration of thin plates resting on Pasternak elastic foundation for different foundation parameters, various modes of vibration and all possible types of classical boundary conditions using the free Galerkin method. Elsewhere, Ketabdari et al. [9] focused on the free vibration analysis of homogeneous and functionally graded skew plates resting on variable Winkler-Pasternak elastic foundation. The elastic foundation was assumed to be a combination of Winkler and Pasternak elastic support. The natural frequency of simply supported functionally graded plates resting on elastic foundation was examined by Gupta et al. [10]. The higher-order shear deformable plate theory of Talha and Singh [11] was used to determine the natural frequencies of simply supported functionally graded square plate resting on elastic foundation. The three-dimensional vibration of a functionally graded sandwich rectangular plate on an elastic foundation with normal boundary conditions was analyzed by Cui et al. [12] using a semi-analytical method based on three-dimensional elasticity theory.

In a separate study, the nonlinear free vibration analysis of functionally graded plates resting on elastic foundation in thermal environment was carried out by Parida and Mohanty [13]. They developed a mathematical model based on a higher-order shear deformation theory using Green-Lagrange type nonlinearity. In their study, Kumar et al. [6] investigated the free vibration behaviour of thin functionally graded rectangular plates by using the dynamic stiffness method (DSM). They adopted the Classical plate theory along with the concept of physical neutral surface of the functionally graded plate to formulate the dynamic stiffness matrix. Zhao-chun et al. [14] more recently assessed the free vibration characteristics of porous functionally graded material (FGM) rectangular plates on a Winkler-Pasternak elastic foundation under the influence of temperature based on the classical thin plate theory and Hamilton principle. To the best of the authors’ knowledge, use of beam analogy method to investigate the free vibration characteristics of FG rectangular plates resting on the Winkler foundation has not yet been investigated.

This study, therefore, assesses the dynamic behaviour of functionally graded rectangular plates resting on Winkler foundation using beam analogy method.

II. THEORETICAL BACKGROUND

The formulation of the exact solution to the governing differential equation of each of the plates studied, development of the characteristic orthogonal shape functions and the fundamental natural frequencies of all round clamped plate (CCCC) with various aspect ratios are presented.

Exact Solution to the Governing Differential Equation

The exact solution to the governing differential equation of the plate has been derived by Ohia et al. [20] as:

$$\omega = \frac{\sqrt{A_1\varphi^4 + B_1\varphi^2 + C_1}}{b^2}\sqrt{\frac{D}{\rho h}} + \sqrt{k} \tag{1}$$

$$H_{b\beta} = \sqrt{A_1\varphi^4 + B_1\varphi^2 + C_1} \tag{2}$$

where

$\omega =$ fundamental natural frequency

$A_1, B_1, C_1 =$ numerical coefficients

$\varphi =$ inverse aspect ratio

D = flexural rigidity of plate

ρ = density of plate

h = thickness of plate

k = reaction coefficient of foundation

$H_{b\beta} =$ Non-dimensional frequency parameter

Characteristic Orthogonal Polynomials (COPs)

Let us consider a rectangular plate of dimensions, a along x and b along y, of uniform thickness shown in Figure 1. If the deflection pattern of the plate along x is represented by a beam strip qualitatively, the beam function along x is taken as F(x). Similarly, the corresponding beam function along y is taken as F(y).

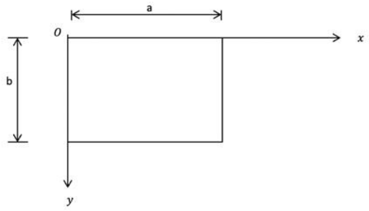


Figure 1: A rectangular plate

Assuming the plate deflections in the form of a series, the solution for prismatic beam of constant stiffness EI and length spanning along x can be written as:

$$w_x = F(x) = \sum_{m=1}^{\infty} X_m x^m \quad (3)$$

and in the y-direction,

$$w_y = F(y) = \sum_{n=1}^{\infty} Y_n y^n \quad (4)$$

Where,

w_x and w_y are plate deflections at point (x,y)

X_m and Y_n are constant parameters in x and y directions respectively

x, y are coordinates of points

m and n are series to infinity limit

F(x) and F(y) are beam functions along x and y directions respectively

Bhat [16] developed a systematic method of constructing the shape function of rectangular plates using the characteristic orthogonal polynomial by assuming the displacement function as a product of two functions: one which is a pure function of x and the other is of y so that,

$w(x,y) = F(x).F(y) = w_x.w_y$

or

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_m x^m Y_n y^n \quad (5)$$

ISSN 2393-8471

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 1, pp: (1-18), Month: April 2024 – September 2024, Available at: www.paperpublications.org

Expressing Equations (3), (4), (5) in the form of non-dimensional parameters, R and Q , Equation (3) becomes

$$w_x = F(\mathrm{x}) = \sum_{m=1}^{\infty} X_m \ (aR)^m \quad = \sum_{m=1}^{\infty} X_m \mathrm{a^m} \mathrm{R}^m \tag{6}$$

In the same manner, substituting y = bQ into Equation (4), we have:

$$w_y = F(\mathrm{y}) = \sum_{n=1}^{\infty} Y_n \ (bQ)^n \quad = \sum_{n=1}^{\infty} Y_n \mathrm{b^n} \mathrm{Q}^n \tag{7}$$

Substituting Equations (5) and (6) into Equation (5) we obtain:

$$\mathrm{w(x,y)} = \sum_{m=1}^{\infty} \sum_{\mathrm{n}=1}^{\infty} X_m \mathrm{a^m} \mathrm{R}^m \ Y_n \mathrm{b^n} \mathrm{Q}^n \tag{8}$$

or

$$\mathrm{w(x,y)} = \sum_{m=1}^{\infty} \sum_{\mathrm{n}=1}^{\infty} A_m \mathrm{R^m} B_n \ \mathrm{Q^n} \tag{9}$$

where

A_m and B_n are coefficients that are to be determined from the boundary conditions at the edges of the plate.

The equation of an orthotropic plate in free vibration is a fourth order differential, the density of the plate being constant. Therefore, m and n in Equation (9) must be equal to 4, Onyeyili [17]. Expanding Equations (7), (8) and (9) to 4th order power series, we obtain

$$w_x = F(\mathrm{x}) = \sum_{m=1}^{4} A_m \mathrm{R}^m = \mathrm{A_0 + A_1R + A_2R^2 + A_3R^3 + A_4R^4} \tag{10}$$

$$\mathrm{w_y} = \mathrm{F(y)} = \sum_{\mathrm{n}=1}^{4} \mathrm{B_nQ^n} \quad = \mathrm{B_0 + B_1Q + B_2Q^2 + B_3Q^3 + B_4Q^4} \tag{11}$$

$$\mathrm{w(x,y)} = \sum_{\mathrm{m}=1}^{\infty} \sum_{n=1}^{\infty} A_m \mathrm{R}^m B_n \mathrm{Q^n} \ = F(\mathrm{x}).\mathrm{G(y)}$$

$$\mathrm{w(x,y) = (A_0 + A_1R + A_2R^2 + A_3R^3 + A_4R^4)(B_0 + B_1Q + B_2Q^2 + B_3Q^3 + B_4Q^4)} \tag{12}$$

The bending moments of plate in x and y directions are given as:

$$\mathrm{M_x} = \frac{-\mathrm{D_x}\, \partial^2 \mathrm{w}}{\partial \mathrm{x}^2} \tag{13}$$

$$\mathrm{M_y} = \frac{-\mathrm{D_y}\, \partial^2 \mathrm{w}}{\partial \mathrm{y}^2} \tag{14}$$

where D_x and D_y are flexural rigidities of the plate in the x and y directions.

Substituting into Equations (13) and (14) w_x and w_y from Equations (10) and (11), Mx and My can be non-dimensionalized into the following expression

$$\mathrm{M_x} = (2A_2 + 6A_3\mathrm{R} + 12A_4\mathrm{R}^2)\frac{-\mathrm{Dx}}{\mathrm{a}^2} \tag{15}$$

and

$$\mathrm{M_y} = (2B_2 + 6B_3\mathrm{Q} + 12B_4\mathrm{Q}^2)\frac{-\mathrm{Dy}}{\mathrm{b}^2} \tag{16}$$

Equations (10), (11), (15), and (16) are used to obtain the displacement functions of the plate.

Boundary Conditions

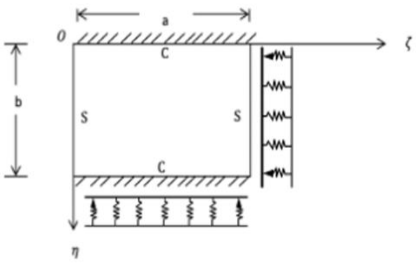


Figure 2: Plan view of SCSC plate on Winkler foundation

Consider the plate to be simply supported on two opposite side and clamped on the other two sides at y = 0 and y = b. The boundary conditions for such a type of mixed edges is,

$$w|_{y=b} = 0 \quad (17)$$

$$\frac{\partial w}{\partial x}|_{y=b} = 0 \quad (18)$$

On the simply supported edges parallel to the 0 axis the boundary condition at x = 0 and x = a

$$w|_{x=a} = 0 \quad (19)$$

$$M_x|_{x=a} = -D\left[\frac{\partial^2 w}{\partial x^2} + \mu\,\frac{\partial^2 w}{\partial y^2}\right]_{x=a} = 0 \quad (20)$$

Development of Shape Function for SCSC Plate

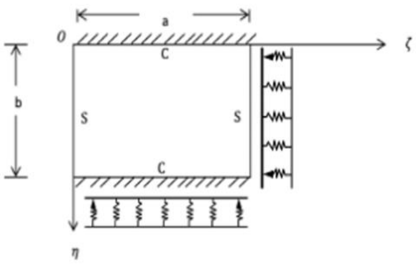


Figure 3: Plate clamped on two opposite edges and simply supported on two other opposite edges (SCSC) resting on Winkler foundation

Boundary conditions

- Deflections at all edges are zero
- Slope at edge η = 0 or 1 is zero
- Moment at edge ζ = 0 or 1 zero

ζ and η represent non-dimensional coordinates in x and y respectively.

For $\zeta -$ directions

At $\zeta = 0$,

Simply Supported edge, $W_x = 0$; $M_x = 0$

From Equation (10)

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 1, pp: (1-18), Month: April 2024 – September 2024, Available at: www.paperpublications.org

$w_x = A_0 + A_1\zeta + A_2\zeta^2 + A_3\zeta^3 + A_4\zeta^4$

At $\zeta = 0$

$$\therefore \quad A_o = 0 \qquad (21)$$

From Equation (15)

$M_x = (2A_2 + 6A_3\zeta + 12A_4\zeta^2)\frac{-Dx}{a^2}$

At $\zeta = 0$

$$M_x = 2A_2\frac{-Dx}{a^2} = 0 \qquad (22)$$

$\therefore A_2 = 0, \quad$ since $\frac{-2Dx}{a^2} \neq 0$

$A_2 = 0 \qquad (23)$

At $\zeta = 1$

$W_x = 0$ and $M_x = 0$

From Equation (10)

$w_x = A_0 + A_1\zeta + A_2\zeta^2 + A_3\zeta^3 + A_4\zeta^4$

At $\zeta = 1$ and bearing in mind that $A_o = 0$ and $A_2 = 0$

$w_x = 0 + A_1 + A_2 + A_3 + A_4 = 0$

$= 0 + A_1 + 0 + A_3 + A_4$

$$A_1 + A_3 + A_4 = 0 \qquad (24)$$

From Equation (15)

$M_x = (2A_2 + 6A_3\zeta + 12A_4\zeta^2)\frac{-Dx}{a^2}$

Since $\frac{-D}{a^2} \neq 0$ and $A_2 = 0$

$M_x = 0 = 6A_3\zeta + 12A_4\zeta^2$

$\Longrightarrow \quad 6A_3 + 12A_4 = 0$

$\Longrightarrow \quad A_3 + 2A_4 = 0$

$\therefore \quad A_3 = -2A_4 \qquad (25)$

Putting Equation (25) into (26), we obtain

$A_1 + (-2A_4) + A_4 = 0$

$\Longrightarrow \quad A_1 - A_4 = 0$

$\therefore \quad A_1 = A_4 \qquad (26)$

Putting the value of $A_0, A_1, A_2,$ and A_3 into Equation (10)

$w_x = A_0 + A_1\zeta + A_2\zeta^2 + A_3\zeta^3 + A_4\zeta^4$

$= A_4\zeta + (-2A_4)\zeta^3 + A_4\zeta^4$

$$w_x = A_4(\zeta - 2\zeta^3 + \zeta^4) \qquad (27)$$

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 1, pp: (1-18), Month: April 2024 – September 2024, Available at: www.paperpublications.org

For $\eta - direction$

At $Q = 0$

Clamped edge, $W_x = 0$, $\frac{\partial w_y}{\partial y} = 0$

From Equation (11),

$$w_y = B_0 + B_1\eta + B_2\eta^2 + B_3\eta^3 + B_4\eta^4$$

At $Q = 0$

$$W_y = B_0 = 0$$

$$\therefore B_0 = 0 \quad (28)$$

Again, from Equation (11)

$$w_y = B_0 + B_1\eta + B_2\eta^2 + B_3\eta^3 + B_4\eta^4$$

$$\frac{\partial w_y}{\partial y} = \frac{\partial w_y}{b\partial\eta} = \frac{1}{b}(B_1 + 2B_2\eta + 3B_3\eta^2 + 4B_4\eta^3) \quad (29)$$

At $\eta = 0$

$$\frac{\partial w_y}{\partial_y} = 0 = \frac{1}{b}B_1 = 0$$

$$since\ \frac{1}{b} \neq 0$$

$$B_1 = 0 \quad (30)$$

At $\eta = 1$

$W_y = 0$, $\frac{\partial W}{\partial y} = 0$

$$w_y = B_0 + B_1\eta + B_2\eta^2 + B_3\eta^3 + B_4\eta^4$$

At $\eta = 1$, bearing in mind that $B_0 = 0$ and $B_1 = 0$

$$w_y = B_2 + B_3 + B_4 = 0$$

$$\Longrightarrow B_2 + B_3 + B_4 = 0 \quad (31)$$

$$B_2 = -(B_3 + B_4) \quad (32)$$

Recall that

$$\frac{\partial w_y}{\partial y} = \frac{1}{b}(B_1 + 2B_2\eta + 3B_3\eta^2 + 4B_4\eta^3)$$

At $\eta = 1$, bearing in mind that $B_1 = 0$

$$\frac{\partial w_y}{\partial_y} = \frac{1}{b}(2B_2 + 3B_3 + 4B_4) = 0$$

$$since\ \frac{1}{b} \neq 0$$

$$2B_2 + 3B_3 + 4B_4 = 0 \quad (33)$$

Putting the value of B_2 of Equation (32) into Equation (33), we obtain,

$$-2(B_3 + B_4) + 3B_3 + 4B_4 = 0$$

ISSN 2393-8471

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 1, pp: (1-18), Month: April 2024 – September 2024, Available at: www.paperpublications.org

$\Longrightarrow B_3 + 2B_4 = 0$

$$\therefore\ B_3 = -2B_4 \tag{34}$$

From Equation (32),

$B_2 = -(B_3 + B_4)$

Substituting $B_3 = -2B_4$ into $B_2 = -(B_3 + B_4)$ we have

$B_2 = -(-2B_4 + B_4)$

$B_2 = -(-B_4)$

$$B_2 = B_4 \tag{35}$$

Putting the expression of $B_0, B_1, B_2 and\ B_3$ into Equation (11),

$\mathrm{w_y} = \mathrm{B_0 + B_1\eta + B_2\eta^2 + B_3\eta^3 + B_4\eta^4}$

$= B_4\eta^2 + (-2B_4)\eta^3 + B_4\eta^4$

$$\mathrm{w_y} = B_4(\eta^2 - 2\eta^3 + \eta^4) \tag{36}$$

Multiplying Equations (27) and (36), we obtain the displacement function for a rectangular plate clamped on two opposite long edges and simply supported on the other two opposite short edge in the form

$W(x, y) = \mathrm{F}(\zeta) * G(\eta) = w_x * w_y$

$= A_4(\zeta - 2\zeta^3 + \zeta^4) * B_4(\eta^2 - 2\eta^3 + \eta^4)$

$= A_4B_4(\zeta - 2\zeta^3 + \zeta^4)(\eta^2 - 2\eta^3 + \eta^4)$

$$W(x, y) = W(\zeta, \eta) = K(\zeta - 2\zeta^3 + \zeta^4)(\eta^2 - 2\eta^3 + \eta^4) \tag{37}$$

Development of Fundamental Natural Frequency Expression of SCSC Plate for Free Vibration

w(x, y) = w (ζ , η) = kSp

where: k = deflection constant

Sp = a polynomial in ζ and η

From Equation (37) we have

$$w(\zeta, \eta) = S_p = (\zeta - 2\zeta^3 + \zeta^4)(\eta^2 - 2.5\eta^3 + \eta^4) \tag{38}$$

$$\frac{\partial^4 S_p}{\partial \zeta^4} = 24(\eta^2 - 2\eta^3 + \eta^4)$$

$$\frac{\partial^4 S_p}{\partial \eta^4} = 24(\zeta - 2\zeta^3 + \zeta^4)$$

$$\frac{\partial^4 S_p}{\partial \zeta^2 \partial \eta^2} = (-12\zeta + 12\zeta^2)(2 - 12\eta + 12\eta^2)$$

$$K_2 = \frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial \zeta^4}\right) + \frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial \zeta^2 \partial \eta^2}\right) + \left(\frac{\partial^4 S_p}{\partial \eta^4}\right)$$

$$= \frac{1}{\beta^4}24(\eta^2 - 2\eta^3 + \eta^4) + \frac{2}{\beta^2}(-12\zeta + 12\zeta^2)(2 - 12\eta + 12\eta^2) + 24(\zeta - 2\zeta^3 + \zeta^4)$$

But

$$\left(\frac{\partial^4 S_p}{\partial \zeta^4}\right) S_p = 24(\eta^2 - 2\eta^3 + \eta^4)(\zeta - 2\zeta^3 + \zeta^4)(\eta^2 - 2.5\eta^3 + \eta^4)$$

ISSN 2393-8471

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 1, pp: (1-18), Month: April 2024 – September 2024, Available at: www.paperpublications.org

$$= 24(\zeta - 2\zeta^3 + \zeta^4)(\eta^4 - 4\eta^5 + 6\eta^6 - 4\eta^7 + \eta^8)$$

$$\left(\frac{\partial^4 S_p}{\partial \eta^4}\right) S_p = 24(\zeta - 2\zeta^3 + \zeta^4)(\zeta - 2\zeta^3 + \zeta^4)(\eta^2 - 2.5\eta^3 + \eta^4)$$
$$= 24(\eta^2 - 2\eta^3 + \eta^4)(\zeta^2 - 4\zeta^4 + 2\zeta^5 + 4\zeta^6 - 4\zeta^7 + \zeta^8)$$

$$\left(\frac{\partial^4 S_p}{\partial \zeta^2 \partial \eta^2}\right) S_p = (-12\zeta + 12\zeta^2)(2 - 12\eta + 12\eta^2)(\zeta - 2\zeta^3 + \zeta^4)(\eta^2 - 2.5\eta^3 + \eta^4)$$
$$= (-12\zeta^2 + 12\zeta^3 + 24\zeta^4 - 36\zeta^5 + 12\zeta^6)(2\eta^2 - 16\eta^3 + 38\eta^4 - 36\eta^5 + 12\eta^6)$$

Recall that

$$S_p = (\zeta - 2\zeta^3 + \zeta^4)(\eta^2 - 2.5\eta^3 + \eta^4)$$

Therefore

$$S_p{}^2 = (\zeta^2 - 4\zeta^4 + 2\zeta^5 + 4\zeta^6 - 4\zeta^7 + \zeta^8)(\eta^4 - 4\eta^5 + 6\eta^6 - 4\eta^7 + \eta^8) \quad (39)$$

Now

$$\int_0^1 \int_0^1 \left(\frac{\partial^4 S_p}{\partial \zeta^4}\right) S_p \, \partial\zeta \partial\eta =$$

$$\int_0^1 \int_0^1 24(\zeta - 2\zeta^3 + \zeta^4)(\eta^4 - 4\eta^5 + 6\eta^6 - 4\eta^7 + \eta^8) \, \partial\zeta \partial\eta$$

$$= 24\left(\frac{1}{2} - \frac{2}{4} + \frac{1}{5}\right)\left(\frac{1}{5} - \frac{4}{6} + \frac{6}{7} - \frac{4}{8} + \frac{1}{9}\right)$$

$$= 0.007619048$$

Therefore,

$$\int_0^1 \int_0^1 \left(\frac{\partial^4 S_p}{\partial \zeta^4}\right) S_p \, \partial\zeta \partial\eta = 0.007619048 \frac{1}{\beta^4}$$

$$\int_0^1 \int_0^1 \left(\frac{\partial^4 S_p}{\partial \zeta^2 \partial \eta^2}\right) S_p \, \partial\zeta \partial\eta$$

$$= (-12\zeta^2 + 12\zeta^3 + 24\zeta^4 - 36\zeta^5 + 12\zeta^6)(2\eta^2 - 16\eta^3 + 38\eta^4 - 36\eta^5 + 12\eta^6) \, \partial\zeta \partial\eta$$

$$= \left(\frac{-12}{2} + \frac{12}{4} + \frac{24}{5} - \frac{36}{6} + \frac{12}{7}\right)\left(\frac{2}{3} - \frac{16}{4} + \frac{38}{5} - \frac{36}{6} + \frac{12}{7}\right)$$

$$= 0.009251701$$

$$\int_0^1 \int_0^1 \frac{2}{\beta^2} \left(\frac{\partial^4 S_p}{\partial \zeta^2 \partial \eta^2}\right) S_p \, \partial\zeta \partial\eta = 0.018503402 \frac{1}{\beta^2}$$

$$\int_0^1 \int_0^1 \left(\frac{\partial^4 S_p}{\partial \eta^4}\right) S_p \, \partial\zeta \partial\eta = 0.039365079$$

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 1, pp: (1-18), Month: April 2024 – September 2024, Available at: www.paperpublications.org

$$C_2 = \int_0^1\int_0^1 (K_2 S_p)\,\partial\zeta\partial\eta$$

$$C_2 = \int_0^1\int_0^1 \left(\frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial\eta^4}\right) + \frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial\zeta^2\partial\eta^2}\right) + \left(\frac{\partial^4 S_p}{\partial\eta^4}\right)\right) S_p\partial\zeta\partial\eta$$

$$= \int_0^1\int_0^1 \frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial\eta^4}\right) S_p\partial\zeta\partial\eta + \int_0^1\int_0^1 \frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial\zeta^2\partial\eta^2}\right) S_p\partial\zeta\partial\eta + \int_0^1\int_0^1 \left(\frac{\partial^4 S_p}{\partial\eta^4}\right) S_p\partial\zeta\partial\eta$$

$$C_2 = 0.007619048\frac{1}{\beta^4} + 0.018503402\frac{1}{\beta^2} + 0.039365079$$

$$\int_0^1\int_0^1 S_p{}^2\,\partial\zeta\partial\eta = \int_0^1\int_0^1 (\zeta^2 - 4\zeta^4 + 2\zeta^5 + 4\zeta^6 - 4\zeta^7 + \zeta^8)(2.25\eta^4 - 7.5\eta^5 + 9.25\eta^6 - 5\eta^7 + \eta^8)\,\partial\zeta\partial\eta$$

$$= \left(\frac{1}{3} - \frac{4}{5} + \frac{1}{3} - \frac{4}{7} - \frac{4}{8} + \frac{1}{9}\right)\left(\frac{1}{5} - \frac{4}{6} + \frac{6}{7} - \frac{4}{8} + \frac{1}{9}\right)$$

$$= 0.000078105$$

$$B_2 = \int_0^1\int_0^1 S_p{}^2\,\partial\zeta\partial\eta = 0.000078105$$

$$\frac{C_2}{B_2} = \frac{0.007619048\frac{1}{\beta^4} + 0.018503402\frac{1}{\beta^2} + 0.039365079}{0.000078105}$$

$$= \frac{97.54868446}{\beta^4} + \frac{236.9041931}{\beta^2} + 504.0020357$$

From Equation (1)

$$\omega = \frac{\sqrt{\frac{97.54868446}{\beta^4} + \frac{236.9041931}{\beta^2} + 504.0020357}}{b^2}\sqrt{\frac{D}{\rho h}} + \sqrt{k} \qquad (40)$$

Comparing Equations (1) and (40) we have

$$H_{b\beta} = \sqrt{\frac{97.54868446}{\beta^4} + \frac{236.9041931}{\beta^2} + 504.0020357}$$

From Equation (2) we have

$$H_{b\beta} = \sqrt{97.54868446\varphi^4 + 236.9041931\varphi^2 + 504.0020357} \qquad (41)$$

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 1, pp: (1-18), Month: April 2024 – September 2024, Available at: www.paperpublications.org

Functionally Graded Plate

A functionally graded plate with length *a*, width *b* and a uniform thickness *h* is considered. The geometry of the plate and the coordinate system are shown in Figure 4.

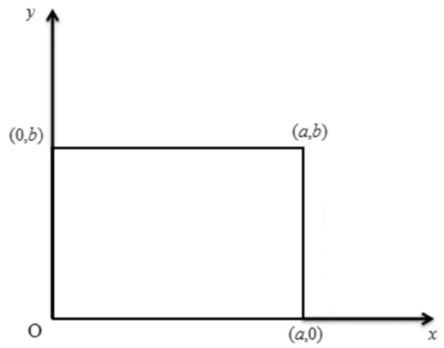


Figure 4: A typical FG rectangular plate element in Cartesian coordinates

It is assumed that the material properties of the FG plate vary smoothly through the thickness. Based on the volume fraction of the constituent material, the Young's modulus and density of FG plate can be written as functions of thickness coordinate, z, as follows (Birman and Byrd [19]):

$$E(z) = (E_c - E_m)\left(\frac{z}{h}+\frac{1}{2}\right)^n + E_m \tag{42}$$

$$\rho(z) = (\rho_c - \rho_m)\left(\frac{z}{h}+\frac{1}{2}\right)^n + \rho_m \tag{43}$$

where *n* is the power law index of the FG rectangular plate, the subscripts m and c show the metal and ceramic surfaces, respectively. Due to the small variations of the Poisson's ratio, ν, it is assumed to be constant (Chakraverty and Pradhan [3]).

According to this distribution, the bottom surface ($z = -\frac{h}{2}$) of FG plate pure metal, whereas the top surface ($z = \frac{h}{2}$) is pure ceramic. The stiffness coefficient is (Chakraverty and Pradhan [3]):

$$D = D_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11}\, Z^2 dz \tag{44}$$

$$= \int_{\frac{-h}{2}}^{\frac{h}{2}} \frac{E(z)}{1-\nu^2} Z^2 dz \tag{45}$$

$$= \frac{1}{1-\nu^2}\int_{\frac{-h}{2}}^{\frac{h}{2}} E(z)\, Z^2 dz \tag{46}$$

$$= \frac{1}{1-\nu^2}\int_{\frac{-h}{2}}^{\frac{h}{2}} \left\{(E_c - E_m)\left(\frac{z}{h}+\frac{1}{2}\right)^n + E_m\right\} Z^2 dz \tag{47}$$

$$= \frac{1}{1-\nu^2}\int_{\frac{-h}{2}}^{\frac{h}{2}} \left\{(E_c - E_m)\left(\frac{z}{h}+\frac{1}{2}\right)^n\right\} Z^2 dz + \int_{\frac{-h}{2}}^{\frac{h}{2}} E_m Z^2 dz \tag{48}$$

$$= \frac{(E_c - E_m)h^3}{1-\nu^2}\left\{\frac{1}{n+3} - \frac{1}{n+2} + \frac{1}{4(n+1)}\right\} + \frac{E_m h^3}{12(1-\nu^2)} \tag{49}$$

While the inertia coefficient is (Chakraverty and Pradhan [3]):

$$I_0 = \rho h = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho(z)\ dz \qquad (50)$$

$$= \frac{(\rho_c - \rho_m)h}{n+1} + \rho_m h \qquad (51)$$

Material Properties of the FGM constituents

An Al/Al_2O_3 functionally graded plate which is composed of aluminum (as metal) and alumina (as ceramic) is considered. The Young's modulus and density of aluminum are E_m = 70 GPa and ρ_m = 2700 kg/m^3, respectively, and that of alumina are E_c = 380 GPa and ρ_c = 3800 kg/m^3, respectively. The Poisson ratio of the plate is assumed to be constant through the thickness and equal to 0.3.

Table 1: Material Properties of the FGM constituents

Properties	Unit	Aluminum (Al)	Alumina (Al_2O_3)
E	GPa	70	380
ρ	Kg/m^3	2700	3800
ν	-	0.3	0.3

III. RESULTS AND DISCUSSION

The results obtained from the preceding section are highlighted here. An Al/Al_2O_3 functionally graded plate which is composed of aluminum (as metal) and alumina (as ceramic) is considered. The Young's modulus and density of aluminum are E_m = 70 GPa and ρ_m = 2700 kg/m^3, respectively, and that of alumina are E_c = 380 GPa and ρ_c = 3800 kg/m^3, respectively. The Poisson ratio of the plate is assumed to be constant through the thickness and equal to 0.3.

The expression for the fundamental natural frequencies of the plate is given as

$$\omega = \frac{\sqrt{A_1\varphi^4 + B_1\varphi^2 + C_1}}{b^2}\sqrt{\frac{D}{\rho h}} + \sqrt{k}$$

The equivalent Winkler parameter is defined as

$$k = \frac{K_w b^4}{D}$$

While the natural frequency equation for the SCSC plate in terms of φ and b is

$$H_{b\beta} = \sqrt{97.54868446\varphi^4 + 236.9041931\varphi^2 + 504.0020357}$$

Table 2 shows the non-dimensional natural frequencies $H_{b\beta}$ for isotropic SCSC plate with various aspect ratios $\beta = \frac{b}{a}$

Table 2: Non-dimensional natural frequencies $H_{b\beta}$ for isotropic SCSC plate with various aspect ratios $\beta = \frac{b}{a}$

k_w	k_s	Aspect Ratio $\beta = \frac{b}{a}$	$H_{b\beta 1}$
0	0	0.1	22.503
		0.2	22.664
		0.3	22.937
		0.4	23.332
		0.5	23.861
		0.6	24.534

		0.7	25.367
		0.8	26.374
		0.9	27.566
		1.0	28.958
100	0	0.1	25.377
		0.2	25.538
		0.3	25.811
		0.4	26.206
		0.5	26.735
		0.6	27.408
		0.7	28.241
		0.8	29.248
		0.9	30.440
		1.0	31.832

In Table 2, the comparison of the natural frequencies of SCSC FG plate with those reported by Chakraverty and Pradhan [3], Hosseini- Hashemi et al. [5], Baferani et al. [18], Bahmyari et. al. [8], Parida and Mohanty [13], for various aspect ratios are presented.

Table 3: Comparison of non-dimensional frequency parameters $H_{b\beta}$ for SCSC plates for various aspect ratios $\beta = \frac{b}{a}$

		Non-Dimensional Frequency Parameter					
(K_W, K_S)	Aspect Ratio $\beta = \frac{b}{a}$	Present study	Chakraverty and Pradhan [3]	Hosseini-Hashemi et al. [5]	Baferani et al. [18]	Bahmyari et. al. [8]	Parida and Mohanty [13]
(0, 0)	0.2	22.664	22.593	-	-	-	-
	0.5	23.861	23.816	-	-	-	-
	1.0	28.958	28.951	28.944	28.944	29.003	28.995
	2.0	54.885	54.743	-	-	-	-
(100, 0)	1.0	31.832	30.629	30.623	30.623	-	30.672

It can be deduced from Table 3 that the Winkler foundation parameter has a dominant influence on the frequencies of plates on elastic foundation. Without considering the effect of Winkler elastic foundation, an increase in aspect ratios leads to increase in frequency parameters.

Table 4: The frequency parameters of SCSC FG rectangular plates with different *n* and aspect ratios, ($K_w = 0$)

Power-law exponent *n*	Aspect ratio (*b/a*)	Present study	Chakraverty and Pradhan [3]
0	0.2	22.664	22.593
	0.5	23.861	23.816
	1.0	28.958	28.951
	2.0	54.885	54.743
0.2	0.2	20.305	21.138
	0.5	21.377	22.282
	1.0	25.943	27.087
	2.0	49.171	51.218
0.5	0.2	19.049	19.843
	0.5	20.055	20.918
	1.0	24.339	25.428
	2.0	46.131	48.081

1.0	0.2	18.045	18.798
	0.5	18.998	19.816
	1.0	23.056	24.089
	2.0	43.699	45.549
2.0	0.2	17.241	17.969
	0.5	18.151	18.942
	1.0	22.028	23.027
	2.0	41.751	43.541

Table 4 shows the non-dimensional frequencies of SCSC FG rectangular plates with different power law exponents, *n* and aspect ratios. It is clear that frequency parameters are increasing with increase in aspect ratios for a given power-law index. It is also noticeable that the frequencies are decreasing with increase in power-law indices for a given aspect ratio.

Table 5: Frequency parameters of FG rectangular plate ($\frac{b}{a} = 2,\ n = 1$) with different elastic moduli (k_w)

BCs	k_w	Frequency parameter
SCSC	0	54.885
	100	57.082

In Table 5, the non-dimensional frequency parameters of SCSC FG plate with aspect ratio $\frac{b}{a} = 2$, and power law index n = 0 are compared with those obtained by Chakraverty and Pradhan [3]. An introduction of a Winkler elastic parameter (k_w = 100) increases the frequency parameter of SCSC plate.

Table 6: The frequency parameters of square SCSC FG Al/Al$_2$O$_3$ plates with different power-law indices (*n*) and k_w = 100

n	Sources	Frequency parameters
0	Present study	30.656
	Chakraverty and Pradhan [3]	30.629
0.2	Present study	29.325
	Chakraverty and Pradhan [3]	28.729
0.5	Present study	27.797
	Chakraverty and Pradhan [3]	27.046
2.0	Present study	25.693
	Chakraverty and Pradhan [3]	24.656

The effect of power law index on the frequency of vibration of SCSC FG plate resting on Winkler elastic foundation is very interesting. As it can be seen in Table 6, the increase in power law index decreases the frequency parameters of the plate.

Table 7: Frequency parameters of SCSC FG Al/Al$_2$O$_3$ plates with different aspect ratios ($\frac{b}{a}$) (n = 1, k_w = 100)

$\frac{b}{a}$	Sources	Frequency parameters
0.2	Present study	21.600
	Chakraverty and Pradhan [3]	20.732
0.5	Present study	22.553
	Chakraverty and Pradhan [3]	21.674
1.0	Present study	26.611
	Chakraverty and Pradhan [3]	25.702
2.0	Present study	47.254
	Chakraverty and Pradhan [3]	46.696

In Table 7, the comparison of the frequency parameters of SCSC FG plate with those reported by Chakraverty and Pradhan [3] using Rayleigh-Ritz method is presented for different aspect ratios. It is observed that with increase in aspect ratios, the frequency parameters increase.

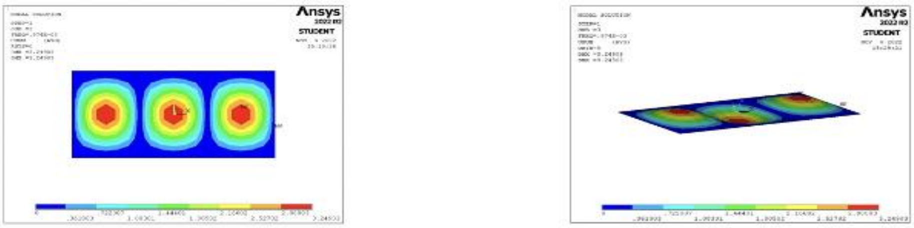
ISSN 2393-8471

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 1, pp: (1-18), Month: April 2024 – September 2024, Available at: www.paperpublications.org

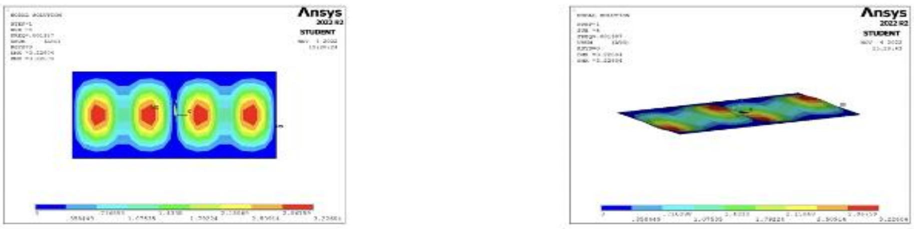
In Figures 5 and 6, the mode shapes of the plate for aspect ratios 0.5 and 1, respectively, are shown.

1st vibration mode

2nd vibration mode



3rd vibration mode



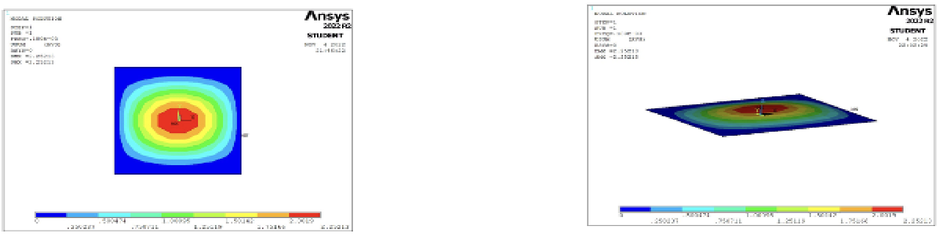
4th vibration mode

5th vibration mode

Figure 5: First five mode shapes of SCSC plates ($\beta = 0.5$)

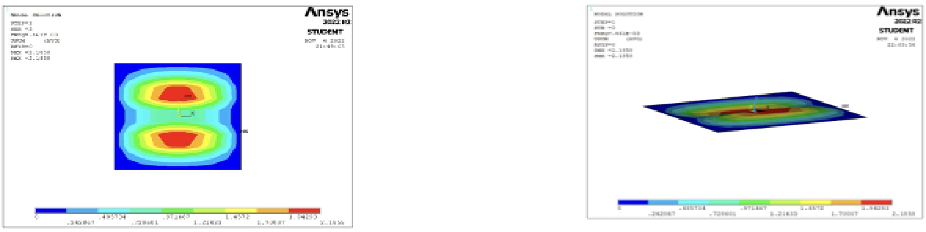
ISSN 2393-8471

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 1, pp: (1-18), Month: April 2024 – September 2024, Available at: www.paperpublications.org



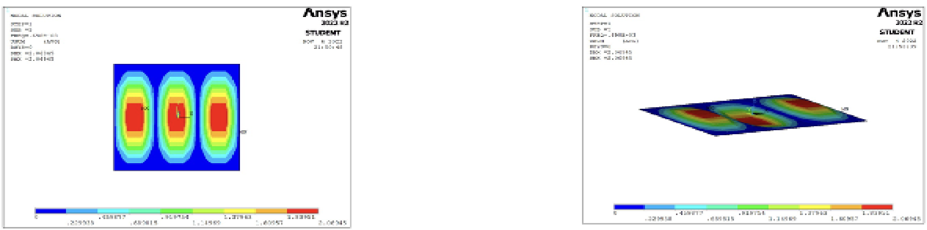
1st vibration mode

2nd vibration mode



3rd vibration mode

4th vibration mode



5th vibration mode

Figure 6: First five mode shapes of SCSC plates ($\beta = 1$)

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 1, pp: (1-18), Month: April 2024 – September 2024, Available at: www.paperpublications.org

IV. CONCLUSION

In this work, integral calculus has been applied to the beam analogy method for the evaluation of the non-dimensional frequency parameters of isotropic FG rectangular plates resting on Winkler elastic foundation. The fundamental assumptions of linear, elastic, small-deflection theory of bending for thin plates due to Kirchhoff are taken into consideration. The Winkler foundation parameter has a dominant influence on the frequencies of plates on elastic foundation. It is evident that adding an elastic foundation increases the non-dimensional frequency parameter of the plates. Without considering the effect of Winkler elastic foundation, an increase in aspect ratios also leads to increase in frequency parameters. It is also observed that an increase in power law index decreases the frequency parameters of the plate. Based on the results obtained, it can be concluded that the model developed present an efficient technique for the evaluation and prediction of the non-dimensional frequency parameters of FG plates resting on Winkler elastic foundation.

REFERENCES

[1] Hsu, M. (2006). Vibrating characteristics of rectangular plates resting on elastic foundations and carrying any number of sprung masses. *International Journal of Applied Science and Engineering*, 4(1), 83-89.

[2] Li, R., Zhong, Y., & Li, M. (2013). Analytic bending solutions of free rectangular thin plates resting on elastic foundations by a new symplectic superposition method. *Proceedings of the Royal Society A*, *469*, 1-18.

[3] Chakraverty, S., & Pradhan, K. K. (2014). Free vibration of functionally graded thin rectangular plates resting on Winkler elastic foundation with general boundary conditions using Rayleigh–Ritz method. *International Journal of Applied Mechanics*, *6*(4), 1-37.

[4] Ramu, I., & Mohanty, S. C. (2015). Free vibration and dynamic stability of functionally graded material plates on elastic foundation. *Defence Science Journal*, 65(3), 245-251.

[5] Hosseini-Hashemi, S., Rokni Damavandi Taher, H., Akhavan, H., & Omidi, M. (2010). Free vibration of functionally graded rectangular plates using first-order shear deformation plate theory. *Applied Mathematical Modelling*, *34*, 1276-1291.

[6] Kumar, S., Ranjan, V., & Jana, P. (2018). Free vibration analysis of thin functionally graded rectangular plates using the dynamic stiffness method. *Composite Structures*, *197*, 39–53.

[7] Sayyad, A. S., & Ghugal, Y. M. (2015). On the free vibration analysis of laminated composite and sandwich plates: a review of recent literature with some numerical results, *Composite Structures*, *129*, 177–201.

[8] Bahmyari, E., Banatehrani, M. M., Ahmadi, M., & Bahmyari, M. (2013). Vibration analysis of thin plates resting on Pasternak foundations by element free Galerkin method. *Shock and Vibration*, 20, 309-326.

[9] Ketabdari, M. J., Allahverdi, A., Boreyri, S., & Ardestani, M. F. (2016). Free vibration analysis of homogeneous and FGM skew plates resting on variable Winkler-Pasternak elastic foundation. *Mechanics & Industry*, 17, 107.

[10] Gupta, A., Talha, M., & Chaudhari, V. K. (2016). Natural frequency of functionally graded plates resting on elastic foundation using finite element method. *Procedia Technology*, 23, 163 – 170.

[11] Talha, M., & Singh, B. N. (2015). Stochastic vibration characteristics of finite element modelled functionally gradient plates. *Composite Structures*, *130*, 95-106.

[12] Cui, J., Zhou, T., Ye, R., Gaidai, O., Li, Z., & Tao, S. (2019). Three-dimensional vibration analysis of a functionally graded sandwich rectangular plate resting on an elastic foundation using a semi-analytical method. *Materials*, 12, 3401

[13] Parida, S., & Mohanty, S. C. (2019). Nonlinear free vibration analysis of functionally graded plate resting on elastic foundation in thermal environment using higher-order deformation theory. *Scientia Iranica B*, 26(2), 815-833.

[14] Zhao-chun, T., Wei-bin, W., & Wen-da, Z. (2022). Free vibration analyses of porous FGM rectangular plates on a Winkler-Pasternak elastic foundation considering the temperature effect. *Engineering Mechanics*, 20(20), 1- 11.

[15] Chakraverty, S. (2009), *Vibration of Plates*, New York, CRC.

[16] Bhat, R. B. (1985). Natural frequencies of rectangular plates using characteristic orthogonal polynomials in the Rayleigh– Ritz method. *Journal of Sound and Vibration*, 102(4), 493–499.

[17] Onyeyili, I. O. (2012). Lecture Notes on Advanced Theory of Plates & Shells, FUTO, SPGS.

[18] Baferani, A. H., Saidi, A. R., & Ehteshami, H. (2011). Accurate solution for free vibration analysis of functionally graded rectangular plates resting on elastic foundation. *Composite Structures*, 93, 1842-1853.

[19] Birman and Byrd (2007). Modeling and analysis of functionally graded materials and structures. ASME Journal of Applied Mechanics, 60, 195-216.

[20] Ohia, C. A., Onwuka, D. O., Okere, C. E., & Onwuka, S. U. (2024). Vibration Characteristics of Functionally Graded Fixed Thin Rectangular Plate on Winkler Elastic Foundation Using Beam Analogy Method. *International Journal of Civil and Structural Engineering Research*, 11.